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Growth property of singular solutions of linear partial differential equations in the complex domain in C^{d+1}

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ABSTRACT. Let $P(z,\partial)$ be a linear partial differential operator with coefficients holomophic in a neighbourhood Ω of $z=0$ in C^{d+1}. Consider equation $P(z,\partial)u(z)=f(z)$, where $u(z)$ and $f(z)$ admit singularities on the surface $\{z_0=0\}$. We assume that $|f(z)|\leq A|z_0|^c$ in a region $\Omega(\theta)$ which is sectorial with respect to z_0. The main result of this paper is the following:

There is an exponent γ^* such that for some class of operators if $\forall\varepsilon>0\ \exists C_\varepsilon$ such that $|u(z)|\leq C_\varepsilon\exp(\varepsilon|z_0|^{-\gamma^*})$ in $\Omega(\theta)$, then $|u(z)|\leq C|z_0|^{c'}$ for some constants c' and C.

First we give the notations briefly. The coordinates of C^{d+1} are denoted by $z=(z_0,z_1,\cdots,z_d)=(z_0,z')\in C\times C^d$. $|z|=\max\{|z_i|;\ 0\leq i\leq d\}$ and $|z'|=\max\{|z_i|;\ 1\leq i\leq d\}$. Its dual variables are $\xi=(\xi_0,\xi')=(\xi_0,\xi_1,\cdots,\xi_d)$. N is the set of all nongegative integers $N=\{0,1,2,\cdots\}$. The differentiation is denoted by $\partial_i=\partial/\partial z_i$, and $\partial=(\partial_0,\partial_1,\cdots,\partial_d)=(\partial_0,\partial')$. For a multi-indes $\alpha=(\alpha_0,\alpha')\in N\times N^d$, $|\alpha|=\alpha_0+|\alpha'|=\sum_{i=0}^d\alpha_i$. Define $\partial^\alpha=\prod_{i=0}^d\partial_i^{\alpha_i}$. We denote $\partial'^{\alpha'}=\prod_{i=1}^d\partial_i^{\alpha_i}$ by $\partial^{\alpha'}$.

We define spaces of holomorphic functions in some regions to state the results. Let $\Omega=\Omega_0\times\Omega'$ be a polydisk with $\Omega_0=\{z_0\in C^1;\ |z_0|<R\}$ and $\Omega'=\{z'\in C^d;|z'|<R\}$ for some positive constant R. Put $\Omega_0(\theta)=\{z_0\in\Omega_0-\{0\};\ |\arg z_0|<\theta\}$ and $\Omega(\theta)=\Omega_0(\theta)\times\Omega'$. $\mathcal{O}(\Omega)$ ($\mathcal{O}(\Omega')$, $\mathcal{O}(\Omega(\theta))$) is the set of all holomorphic functions on Ω (*resp.* Ω', $\Omega(\theta)$). $\mathcal{O}(\Omega(\theta))$ contains multi-valued functions, if $\theta>\pi$.

We introduce $\mathcal{O}_{(\kappa)}(\Omega(\theta))$ and $Asy_{\{\kappa\}}(\Omega(\theta))$, which are subspaces of $\mathcal{O}(\Omega(\theta))$ and fundamental function spaces in this paper.

Definition 1. *$\mathcal{O}_{(\kappa)}(\Omega(\theta))$ $(0<\kappa<+\infty)$ is the set of all $u(z)\in\mathcal{O}(\Omega(\theta))$ such that for any $\varepsilon>0$ and any θ' with $0<\theta'<\theta$*

$$|u(z)|\leq C\exp(\varepsilon|z_0|^{-\kappa})\quad z\in\Omega(\theta') \tag{1}$$

holds for a constant $C=C(\varepsilon,\theta')$. We put $\mathcal{O}_{(+\infty)}(\Omega(\theta))=\mathcal{O}(\Omega(\theta))$ for $\kappa=+\infty$.

Definition 2. *$\mathcal{O}_{reg,c}(\Omega(\theta))$ is the set of all $u(z)\in\mathcal{O}(\Omega(\theta))$ such that any θ' with $0<\theta'<\theta$*

$$|u(z)|\leq C|z_0|^c\quad z\in\Omega(\theta') \tag{2}$$

holds for a constant $C=C(\theta')$.

We say that $u(z)\in\mathcal{O}(\Omega(\theta))$ is *slowly increasing* in $\Omega(\theta)$, if $u(z)\in\bigcup_{|c|<+\infty}\mathcal{O}_{reg,c}(\Omega(\theta))$.

Now let $P(z,\partial)$ be an m-th order linear partial differential equation with coeficients in $\mathcal{O}(\Omega)$

$$(3) \qquad P(z,\partial) = \sum_{|\alpha|\le m} a_\alpha(z)\partial^\alpha = \sum_{|\alpha|\le m} z_0^{j_\alpha} b_\alpha(z)\partial^\alpha,$$

where $j_\alpha \in \boldsymbol{N}$ is the valuation of $a_\alpha(z)$ with respect to z_0, $a_\alpha(z) = z_0^{j_\alpha} b_\alpha(z)$. Let us define some quantities for $P(z,\partial)$:

(4)

$$\begin{cases} e_* := \min\{j_\alpha - \alpha_0; \alpha \in \boldsymbol{N}^{d+1}\},\ \Delta = \{\alpha \in \boldsymbol{N}^{d+1}; j_\alpha - \alpha_0 = e_*\} \\ k^* := \max\{|\alpha|;\ \alpha \in \Delta\}. \end{cases}$$

Put

$$(5) \qquad \mathfrak{P}(z,\partial) = \sum_{\alpha\in\Delta} z_0^{j_\alpha} b_{\alpha,0}(z')\partial^\alpha.$$

Let us introduce an index which plays an important role in this paper.

Definition 3. *(Minimal irregularity)*

$$(6) \quad \begin{cases} \gamma^* := \min\{\dfrac{j_\alpha - \alpha_0 - e_*}{|\alpha| - k^*}; \alpha \in \boldsymbol{N}^{d+1},\ |\alpha| > k^*\}, & \textit{if} \quad k^* < m, \\ \gamma^* := \infty, \quad \textit{if} \quad k^* = m. \end{cases}$$

Let us introduce conditions on $P(z,\partial)$.

Condition 0. *If $\alpha = (\alpha_0, \alpha') \in \Delta$, then $\alpha' = (0,0,\cdots,0)$.*

The following condition is more strict than Condition 0.

Condition 1. *$P(z,\partial)$ satisfies Condition 0 and $b_{(k^*,0,0,\cdots,0)}(0) \neq 0$.*

Suppose that $P(z,\partial)$ satisfies Condition 0. Then $\mathfrak{P}(z,\partial)$ is an ordinary differential operator,

$$(7) \qquad \mathfrak{P}(z,\partial) = \sum_{\alpha\in\Delta} z_0^{e_*} b_{\alpha,0}(z') z_0^{\alpha_0}\partial_0^{\alpha_0},$$

and $\{z_0 = 0\}$ is regular singular. Define the indicial polynomial $\chi_P(z',\lambda)$ of $\mathfrak{P}(z,\partial)$,

$$(8) \qquad \chi_P(z',\lambda) := \sum_{\alpha\in\Delta} b_{\alpha,0}(z')\lambda(\lambda-1)\cdots(\lambda-\alpha_0+1).$$

Further suppose that $P(z,\partial)$ satisfies Condition 1. Then $\chi_P(z',\lambda)$ is a polynomial of λ with degree k^* in $\{z;\ |z| \le R\}$. Hence there exist real constants a_0, a_1 and b_0 such that all the roots of $\chi_P(z',\lambda) = 0$ for $|z| \le R$ are contained in $\{\lambda; a_0 \le \Re\lambda \le a_1,\ |\Im\lambda| \le b_0\}$.

Now let us consider

$$(\mathrm{Eq}) \qquad P(z,\partial)u(z) = f(z).$$

We have results concerning the growth properties of solutions of (Eq).

Theorem 4. *Suppose that $P(z,\partial)$ satisfies Condition 1. Let $u(z) \in \mathcal{O}_{(\gamma^*)}(\Omega(\theta))$ be a solution of* (Eq). *Suppose that $f(z) \in \mathcal{O}_{reg,c}(\Omega(\theta))$. Then there is a polydisk U centered at $z = 0$ such that $u(z) \in \mathcal{O}_{reg,c'}(U(\theta))$ for any $c' < \min\{c - e_*, a_0\}$.*

Theorem 5. *Suppose that $P(z,\partial)$ satisfies Condition 0. Let $u(z) \in \mathcal{O}_{(\gamma^*)}(\Omega(\theta))$ be a solution of* (Eq). *Suppose that $f(z) \in \mathcal{O}_{reg,c}(\Omega(\theta))$. Then there is a polydisk U centered at $z = 0$ and a constant c" such that $u(z) \in \mathcal{O}_{reg,c''}(U(\theta))$.*

We show Theorem 4 by constructing a parametrix and Theorem 5 follows from Theorem 4. The proof theorems and the details of this paper will be appeared in the forthcoming paper.

We give some examples satisfying *Condition 1*:
(a). Operators of normal type with respect to ∂_0,

$$\partial_0^{k^*} + \sum_{\alpha_0 < k^*} a_\alpha(z)\partial^\alpha.$$

(b). Operators of Fuchsian type.

(c). Other concrete examples are

$$I_d + z_0^2\partial_0 + z_0\partial_1^2, \quad z_0\partial_0^2 + a(z)\partial_0 + \partial_1^3.$$

The present paper follows Ōuchi [4]. The class of operators considered in [4] was more strict than that of this paper. The main Theorem in [4] was the following:

If $u(z)$ grows at most some exponential order near $z_0 = 0$, that is, for any $\varepsilon > 0$ $|u(z)| \le C_\epsilon \exp(\varepsilon|z_0|^{-\gamma^*})$ near $z_0 = 0$, and if $f(z)$ hehaves asymptotically $f(z) \sim \sum_{n=0}^{+\infty} f_n(z')z_0^n$ as $z_0 \to 0$ in a sectorial region $\Omega(\theta)$, where $|f_n(z')| \le AB^n\Gamma(n/\gamma^* + 1)$, then $u(z)$ has also the asymptotic expansion like $f(z)$ as z_0 tends to 0.

It was an extension of the main result of [1] and [2]. But in the present paper we treat a wider class of operators which contains that of [4]. So even if $f(z)$ has a Gevrey type asymptotic expansion, $u(z)$ does not always have. Hence, Theorem 4 in this paper is somewhat different. Roughly speaking,

if $u(z)$ grows at most some exponential order near $\{z_0 = 0\}$, and if $f(z)$ has the slowly increasing singularities on $\{z_0 = 0\}$, then the growth order of singularities of $u(z)$ are also slowly increasing.

We can show the results in [4], by using Theorem 4.

References

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